

# Kaluza-Klein braneworld cosmology with static internal dimensions

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We investigate the Kaluza-Klein braneworld cosmology from the point of view of observers on the brane. We first generalize the Shiromizu-Maeda-Sasaki (SMS) equations to higher dimensions. As an application, we study a  $(4+n)$ -dimensional brane with  $n$  dimensions compactified on the brane, in a  $(5+n)$ -dimensional bulk. By assuming that the size of the internal space is static, that the bulk energy-momentum tensor can be neglected, we determine the effect of the bulk geometry on the Kaluza-Klein braneworld. Then we derive the effective Friedmann equation on the brane. It turns out that the Friedmann equation explicitly depends on the equation of state, in contrast to the braneworld in a 5-dimensional bulk spacetime. In particular, in a radiation-dominated era, the effective Newton constant depends on the scale factor logarithmically. If we include a pressureless matter on the brane, this dependence disappears after the radiation-matter equality. This may be interpreted as stabilization of the Newton constant by the matter on the brane. Our findings imply that the Kaluza-Klein braneworld cosmology is quite different from the conventional Kaluza-Klein cosmology even at low energy.

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## I. INTRODUCTION

Early universe models are usually motivated by theories trying to describe fundamental physics at high energies. String theory is a leading candidate to describe high energy physics and since it requires 10 dimensions to be consistent, it is natural to consider higher dimensional universes. To reconcile a higher dimensional universe with our empirically 4-dimensional universe, the traditional approach has been to resort to the Kaluza-Klein compactification. Another option, the braneworld scenario, was proposed about a quarter of a century ago [1, 2] and has been revived recently, stimulated by the discovery of D-branes [3]. In particular, Randall and Sundrum (RS) proposed an interesting framework [4, 5], partly inspired by the Horava-Witten model [6]. RS formulated the universe as a domain wall in 5-dimensional Anti deSitter (AdS) spacetime. Much work has been done on its cosmology (see [7, 8, 9, 10] for reviews) and black hole physics [11, 12, 13, 14, 15, 16]. However, the RS framework with codimension one braneworld is insufficient to reach 10 dimensions. To go beyond five dimensions while keeping our spacetime dimensions to four, we need to consider either higher codimensions [17, 18] or Kaluza-Klein compactification on the brane. The former option, i.e. to realize a higher codimension braneworld is difficult due to the strong self-gravity of the brane. In fact, a higher codimension braneworld develops a severe singularity except for codimension two models. As a result, no successful cosmological model is known. Even in the case of codimension two models, it seems almost impossible to construct a consistent cosmological model due to the subtlety of the conical singularity [19, 20, 21, 22]. The latter option, i.e. to consider a Kaluza-Klein cosmology on the brane [23, 24, 25, 26], is our concern in this paper.

One might think it is a trivial task to construct braneworld models with Kaluza-Klein compactification. Unfortunately, it is not so [27]. In the case of the RS model, the bulk geometry is given and static. Hence, the cosmology on the brane is simply due to its motion in the bulk spacetime. In the case of Kaluza-Klein braneworlds, however, the bulk geometry is not known a priori [28, 29, 30]. Moreover, as we require the internal space to be static, we might have to take into account the matter in the bulk, in the form for instance of fluxes. It makes it difficult to solve the bulk geometry in most cases. In general, we have to solve the bulk geometry and the brane motion at the same time and explicit analytical examples are difficult to construct (see e.g. [31] for anisotropic 5D bulk-brane configurations, with a problematics similar in spirit to Kaluza-Klein braneworlds). Although numerical methods seem to be inescapable in general, an analytical approach would be useful even if it is a modest one.

In this paper, we make a first step in the analytical description of Kaluza-Klein braneworlds. Here, we do not intend to solve the bulk geometry. Instead, we use the Shiromizu-Maeda-Sasaki (SMS) equation [32] to analyze the Kaluza-Klein cosmology. Of course, this effective equation cannot be solved without knowing the projected Weyl tensor. Hence, we take the following strategy. We use the staticity of the internal space as a principle to constrain the unknown bulk geometry. We also assume that the bulk matter can be neglected, at least in the vicinity of the branes, in the regimes which we study. Then, we can determine the Friedman equation on the brane. Interestingly, the resultant Friedman equation is found to depend on the equation of state of the matter explicitly. In particular, the

effective Newton constant varies logarithmically at a radiation-dominated stage. Thus the Kaluza-Klein braneworld cosmology appears to be quite different from the conventional Kaluza-Klein cosmology even at low energy.

The organization of this paper is as follows. In section 2, we consider a braneworld model with the bulk matter in general dimensions and derive the effective SMS equations on the brane. In section 3, we apply the SMS equations to a  $(5+n)$ -dimensional Kaluza-Klein braneworld model and derive the effective Friedmann equation on the brane by imposing the stability of the internal space. The final section is devoted to conclusion.

## II. SMS EFFECTIVE EQUATION IN (D+1)-DIMENSIONS

In this section, we derive the effective gravitational equations on the brane for any dimension. To be as general as possible, we also include a bulk energy momentum tensor.

The action we consider is

$$S = \frac{1}{2\kappa^2} \int d^{d+1}x \sqrt{-\tilde{g}} [R - 2\Lambda] - \sigma \int d^d x \sqrt{-g} + S_m; \quad \Lambda = -\frac{d(d-1)}{2\ell^2} \quad (1)$$

where  $\kappa^2$ ,  $\ell$  and  $\sigma$  are the gravitational coupling constant, the scale of the bulk curvature radius and the tension of the brane, respectively. We assume a negative cosmological constant in the bulk. Here,  $S_m$  represents the action for the matter both in the bulk and on the brane. The  $(d+1)$ -dimensional and  $d$ -dimensional metrics are represented by  $\tilde{g}$  and  $g$ , respectively.

We consider a  $d$ -dimensional brane with  $(d-4)$  compactified dimensions. To describe the bulk spacetime, we can use Gaussian Normal coordinates so that the metric takes the form

$$ds^2 = dy^2 + g_{\mu\nu}(y, x^\mu) dx^\mu dx^\nu, \quad (2)$$

and the brane position is  $y = 0$  in this coordinate system. One can deduce the effective equation on the brane following SMS. The extrinsic curvature is defined as

$$K_{\mu\nu} = -\frac{1}{2} \frac{\partial}{\partial y} g_{\mu\nu} \equiv -\frac{1}{2} g_{\mu\nu,y}. \quad (3)$$

Using the extrinsic curvature, we can write down the Einstein equations in  $(d+1)$ -dimensions as

$$^{(d+1)}G^y_y = -\frac{1}{2}R + \frac{1}{2}K^2 - \frac{1}{2}K^{\alpha\beta}K_{\alpha\beta} = \frac{d(d-1)}{2\ell^2} + \kappa^2 T^y_y, \quad (4)$$

$$^{(d+1)}G^y_\mu = -\nabla_\lambda K_\mu^\lambda + \nabla_\mu K = \kappa^2 T^y_\mu, \quad (5)$$

$$^{(d+1)}G^\mu_\nu = G^\mu_\nu + (K^\mu_\nu - \delta^\mu_\nu K)_{,y} - K K^\mu_\nu + \frac{1}{2} \delta^\mu_\nu (K^2 + K^{\alpha\beta}K_{\alpha\beta}) = \frac{d(d-1)}{2\ell^2} \delta^\mu_\nu + \kappa^2 S^\mu_\nu \delta(y) + \kappa^2 T^\mu_\nu, \quad (6)$$

where  $G_{\mu\nu}$  is the  $d$ -dimensional Einstein tensor, and  $T_{\mu\nu}$ ,  $T_{y\mu}$ , and  $T_{yy}$  are the components of the bulk energy momentum tensor. Here  $\nabla_\mu$  denotes the covariant derivative with respect to the metric  $g_{\mu\nu}$ , and  $S_{\mu\nu} = -\sigma g_{\mu\nu} + t_{\mu\nu}$  is the energy momentum tensor on the brane, where  $t_{\mu\nu}$  is the energy momentum tensor of the brane matter other than the tension. Then, the junction conditions are given by

$$[K^\mu_\nu - \delta^\mu_\nu K]_{y=0} = \frac{\kappa^2}{2} (-\sigma \delta^\mu_\nu + t^\mu_\nu), \quad (7)$$

where we have assumed  $Z_2$ -symmetry. Combining Eqs. (4) with (6), we have

$$\begin{aligned} -\frac{1}{d-1} \left( R_{\mu\nu} - \frac{1}{d} g_{\mu\nu} R \right) &= \frac{1}{d-1} [K_{\mu\nu,y} - g_{\mu\nu} K_{,y} - K K_{\mu\nu} + 2K_\mu^\lambda K_{\lambda\nu}] \\ &+ \frac{1}{d(d-1)} g_{\mu\nu} K^2 + \frac{1}{d} g_{\mu\nu} K^{\alpha\beta} K_{\alpha\beta} - \frac{1}{\ell^2} g_{\mu\nu} - \frac{\kappa^2}{d-1} \left( T_{\mu\nu} - \frac{d-2}{d} g_{\mu\nu} T^y_y \right). \end{aligned} \quad (8)$$

The trace of this equation gives

$$K_{,y} = K^{\alpha\beta} K_{\alpha\beta} - \frac{d}{\ell^2} + \kappa^2 \frac{d-2}{d-1} T^y_y - \frac{\kappa^2}{d-1} T^\mu_\mu \quad (9)$$

Also the following components of the Weyl tensor are relevant.

$$C_{y\mu y\nu} = -\frac{1}{d-1} \left( R_{\mu\nu} - \frac{1}{d} g_{\mu\nu} R \right) + \frac{d-2}{d-1} K_{\mu\nu,y} - \frac{d-2}{d(d-1)} g_{\mu\nu} K_{,y} + \frac{d-3}{d-1} K_{\mu}{}^{\lambda} K_{\lambda\nu} \\ + \frac{1}{d-1} K K_{\mu\nu} + \frac{1}{d} g_{\mu\nu} K^{\alpha\beta} K_{\alpha\beta} - \frac{1}{d(d-1)} g_{\mu\nu} K^2 . \quad (10)$$

The above components of the Weyl tensor can be rewritten by using Eqs. (8) and (9) as

$$C_{y\mu y\nu} = K_{\mu\nu,y} - g_{\mu\nu} K_{,y} + K_{\mu}{}^{\lambda} K_{\lambda\nu} + g_{\mu\nu} K^{\alpha\beta} K_{\alpha\beta} - \frac{d-1}{\ell^2} g_{\mu\nu} \\ + \kappa^2 \frac{d-2}{d} g_{\mu\nu} T^y{}_y - \frac{\kappa^2}{d-1} \left( T_{\mu\nu} + \frac{d-2}{d} g_{\mu\nu} T^{\alpha}{}_{\alpha} \right) . \quad (11)$$

Thus off the brane, using these components of the Weyl tensor, Eq. (6) is expressed as

$$G_{\mu\nu} = -C_{y\mu y\nu} - K_{\mu}{}^{\lambda} K_{\lambda\nu} + K K_{\mu\nu} + \frac{1}{2} g_{\mu\nu} K^{\alpha\beta} K_{\alpha\beta} - \frac{1}{2} g_{\mu\nu} K^2 + \frac{(d-1)(d-2)}{2\ell^2} g_{\mu\nu} \\ + \frac{d-2}{d-1} \kappa^2 \left( T_{\mu\nu} - \frac{1}{d} g_{\mu\nu} T^{\alpha}{}_{\alpha} \right) + \frac{d-2}{d} \kappa^2 T^y{}_y g_{\mu\nu} \quad (12)$$

where we stress that the term  $T^{\alpha}{}_{\alpha}$  is the trace defined with respect to the  $d$ -dimensional metric  $g$ , and not the full trace defined with respect to  $\tilde{g}$ . Eliminating the extrinsic curvature by using the junction conditions (7), and assuming the RS type relation

$$\kappa^2 \sigma = \frac{2(d-1)}{\ell} , \quad (13)$$

we finally obtain the  $d$ -dimensional generalization of the SMS equations,

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -E_{\mu\nu} + 8\pi G t_{\mu\nu} + \kappa^4 \pi_{\mu\nu} + \frac{d-2}{d-1} \kappa^2 \left( T_{\mu\nu} - \frac{1}{d} g_{\mu\nu} T^{\alpha}{}_{\alpha} \right) + \frac{d-2}{d} \kappa^2 T^y{}_y g_{\mu\nu} , \quad (14)$$

where we have defined the Newton constant in  $d$ -dimensions by

$$8\pi G = \frac{(d-2)\kappa^2}{2\ell} , \quad (15)$$

the tensor

$$\pi_{\mu\nu} = \frac{1}{4(d-1)} t t_{\mu\nu} - \frac{1}{4} t_{\mu}{}^{\lambda} t_{\lambda\nu} + \frac{1}{8} g_{\mu\nu} t^{\alpha\beta} t_{\alpha\beta} - \frac{1}{8(d-1)} g_{\mu\nu} t^2 , \quad (16)$$

and the projected Weyl tensor

$$E_{\mu\nu} = C_{y\mu y\nu} \big|_{y=0} . \quad (17)$$

These results are also obtained in [33, 34].

From the Bianchi identity satisfied by the Einstein tensor, we can deduce a constraint equation on the tensors that appear on the right hand side of (14)

$$\nabla^{\mu} E_{\mu\nu} = \kappa^4 \nabla^{\mu} \pi_{\mu\nu} + \frac{d-2}{d-1} \kappa^2 \nabla^{\mu} T_{\mu\nu} - \frac{d-2}{d(d-1)} \kappa^2 \nabla_{\nu} T^{\alpha}{}_{\alpha} + \frac{d-2}{d} \kappa^2 \nabla_{\nu} T^y{}_y , \quad (18)$$

where we have assumed the conservation of the energy-momentum tensor for the matter on the brane

$$\nabla^{\mu} t_{\mu\nu} = 0 , \quad (19)$$

i.e. we forbid the possibility of energy exchange between the brane and the bulk (as studied e.g. in [35] in 5D brane cosmology). We also have the conservation law for the bulk matter, which can be decomposed as

$$0 = \partial_y T^y{}_y - K T^y{}_y + K^{\mu}{}_{\nu} T^{\nu}{}_{\mu} + \nabla_{\mu} T^{\mu}{}_y , \quad (20)$$

$$0 = \partial_y T^y{}_{\mu} - K T^y{}_{\mu} + \nabla_{\nu} T^{\nu}{}_{\mu} . \quad (21)$$

Of course, the above equations do not form a closed system, because we do not know  $E_{\mu\nu}$ . In other words, without knowing the bulk geometry, we cannot solve the effective Einstein equations (14). However, we may regard the SMS equation as an initial value equation. Namely, once  $E_{\mu\nu}$  is given from the SMS equation, we can solve the  $(d+1)$ -dimensional Einstein equations along the  $y$ -direction to obtain the bulk geometry. In this picture, unless we impose some conditions on the properties of the spacetime, there will be too many allowed bulk solutions, and most of the solutions will be physically meaningless.

In the next section, we will try to solve the effective Einstein's equations to study the cosmology of Kaluza-Klein braneworlds from the SMS equation, by assuming that the internal dimensions are static and that the bulk energy-momentum tensor can be neglected on the brane. Given these conditions, we will show that, similarly to the cosmology of a codimension 1 brane in an empty 5D bulk [36, 37], the effective Friedmann equations can be solved, up to a constant of integration. However, in contrast to the 5D bulk, we will not give a bulk geometry associated with this cosmology.

### III. KALUZA-KLEIN BRANEWORLD COSMOLOGY

To realize a braneworld in higher dimensions, it seems natural to consider a codimension one brane with internal dimensions that are compactified *à la* Kaluza-Klein, in short a Kaluza-Klein braneworld. Here, for simplicity, we consider  $n$  internal dimensions compactified on a torus. The brane thus represents a  $(4+n)$ -dimensional spacetime embedded in a  $(5+n)$ -dimensional spacetime. Since we wish to study the cosmology, we consider metrics of the form

$$ds^2 = -dt^2 + a^2(t)\delta_{ij}dx^i dx^j + b^2(t)\delta_{\alpha\beta}dz^\alpha dz^\beta, \quad (22)$$

where the  $x^i$  are the three ordinary spatial coordinates and the  $z^\alpha$  are the internal coordinates. For simplicity, we assume that there is a single scale factor  $b$  characterizing the size of the internal dimensions. The scale factor  $a$  is the usual scale factor for the external space.

We can imagine two kinds of matter, the matter in the bulk and on the brane. The bulk matter is important to get a well-behaved geometry in the bulk [39, 40]. However, for simplicity, we suppose here that we can ignore it for the cosmology on the brane. Hereafter, we will consider only the matter on the brane. Note a recent work, where a similar analysis was considered for a 6D Kaluza-Klein brane embedded in a 7D bulk spacetime, and which takes into account a bulk energy-momentum tensor and the possibility of brane-bulk energy exchange [38]. Because of the symmetries, the energy-momentum tensor is restricted to be of the following form,

$$t_{\mu\nu} = (\rho, Pa^2\delta_{ij}, Qb^2\delta_{\alpha\beta}) , \quad (23)$$

where  $\rho$  is the energy density,  $P$  the external pressure and  $Q$  the internal pressure. Similarly, the projected Weyl tensor is of the form

$$E_{\mu\nu} = (e, \chi a^2\delta_{ij}, \xi b^2\delta_{\alpha\beta}) . \quad (24)$$

Moreover, the traceless property of  $E_{\mu\nu}$  implies the relation  $-e + 3\chi + n\xi = 0$ . The components of the quadratic tensor in the energy-momentum tensor (16) are given by

$$\pi_{00} = \frac{1}{8(3+n)} \left[ (n+2)\rho^2 - 3n(P-Q)^2 \right] , \quad (25)$$

$$\pi_{ij} = \frac{1}{8(3+n)} \left[ (n+2)\rho^2 + 2\rho(2P+nQ) + n(P-Q)(P-3Q) \right] a^2\delta_{ij} , \quad (26)$$

$$\pi_{\alpha\beta} = \frac{1}{8(3+n)} \left[ (n+2)\rho^2 + 6\rho P + 2(n-1)\rho Q + 3(n(P-Q)^2 - 2Q^2 + 2PQ) \right] b^2\delta_{\alpha\beta} . \quad (27)$$

Substituting the metric (22) and the tensors (23), (24) and (25-27) in the effective Einstein equations (14), one finds

$$\begin{aligned} 3H_a^2 + 3nH_aH_b + \frac{n(n-1)}{2}H_b^2 &= 8\pi G\rho + \frac{\kappa^4}{8(3+n)} \left[ (n+2)\rho^2 - 3n(P-Q)^2 \right] - e \quad (28) \\ -2\dot{H}_a - 3H_a^2 - 2nH_aH_b - n\dot{H}_b - \frac{n(n+1)}{2}H_b^2 &= 8\pi GP + \frac{\kappa^4}{8(3+n)} \left[ (n+2)\rho^2 \right. \\ &\quad \left. + 2\rho(2P+nQ) + n(P-Q)(P-3Q) \right] - \chi , \quad (29) \\ -3\dot{H}_a - 6H_a^2 - 3(n-1)H_aH_b - (n-1)\dot{H}_b - \frac{n(n-1)}{2}H_b^2 &= 8\pi GQ + \frac{\kappa^4}{8(3+n)} \left[ (n+2)\rho^2 + 6\rho P + 2(n-1)\rho Q \right. \end{aligned}$$

$$+3 \left( n(P-Q)^2 - 2Q^2 + 2PQ \right) ] - \xi . \quad (30)$$

In addition to the above equations, we need the conservation law for the matter (19), which is given here by

$$\dot{\rho} + 3H_a(\rho + P) + nH_b(\rho + Q) = 0 . \quad (31)$$

The constraint equation for the projected Weyl tensor (18) in the absence of the bulk matter can be written as

$$\dot{e} + 3H_a(e + \chi) + nH_b(e + \xi) + \frac{3n}{4(3+n)}\kappa^4(P-Q) \left[ \dot{P} - \dot{Q} + H_a(\rho + P) - H_b(\rho + Q) \right] = 0 , \quad (32)$$

where we have defined the Hubble parameters  $H_a = \dot{a}/a$  and  $H_b = \dot{b}/b$ . In order to integrate explicitly these equations, we will assume simple equations of state for the anisotropic fluid, namely  $P = w\rho$  and  $Q = v\rho$ , where  $w$  and  $v$  are constants. Then the equations (28-30) become

$$3H_a^2 + 3nH_aH_b + \frac{n(n-1)}{2}H_b^2 = 8\pi G\rho + \frac{\kappa^4}{8(3+n)} \{2 + n(1 - 3(v-w)^2)\} \rho^2 - e , \quad (33)$$

$$\begin{aligned} -2\dot{H}_a - 3H_a^2 - 2nH_aH_b - n\dot{H}_b - \frac{n(n+1)}{2}H_b^2 &= 8\pi Gw\rho + \frac{\kappa^4}{8(3+n)} \{2 + 4w + \\ &\quad + n(1 + 2v + 3v^2 - 4vw + w^2)\} \rho^2 - \chi , \end{aligned} \quad (34)$$

$$\begin{aligned} -3\dot{H}_a - 6H_a^2 - 3(n-1)H_aH_b - (n-1)\dot{H}_b - \frac{n(n-1)}{2}H_b^2 &= 8\pi Gv\rho + \frac{\kappa^4}{8(3+n)} \{2 - 2v - 6v^2 + 6w + 6vw + \\ &\quad + n(1 + 2v + 3v^2 - 6vw + 3w^2)\} \rho^2 - \frac{1}{n}(e - 3\chi) \end{aligned} \quad (35)$$

and the conservation law reduces to

$$\dot{\rho} + 3(1+w)H_a\rho + n(1+v)H_b\rho = 0 . \quad (36)$$

The constraint (32) is written as

$$\dot{e} + 3H_a(e + \chi) + H_b((n+1)e - 3\chi) + \frac{3n}{4(3+n)}\kappa^4 [(w-v)^2\rho\dot{\rho} + (H_a(1+w) - H_b(1+v))(w-v)\rho^2] = 0 . \quad (37)$$

Since we do not know  $e$ ,  $\chi$  and  $\xi$ , we cannot solve the above equations. We need to solve the bulk geometry to obtain  $e$ ,  $\chi$  and  $\xi$  in general. Here, instead of solving the bulk geometry, we impose the stability of the internal space, that is, we put  $b = 1$ . By doing so, we may have a singularity in the bulk. However, if we allow the existence of matter in the bulk, it is reasonable to expect that the bulk geometry can be made regular by a suitable choice of the bulk matter. What kind of matter is necessary is a different issue. Here we assume the staticity of the internal space, and seek for an effective Friedman equation on the brane. Under this staticity assumption, Eqs. (33) - (37) reduce to

$$3H_a^2 = 8\pi G\rho + \frac{\kappa^4}{8(3+n)} \{2 + n(1 - 3(v-w)^2)\} \rho^2 - e , \quad (38)$$

$$-2\dot{H}_a - 3H_a^2 = 8\pi Gw\rho + \frac{\kappa^4}{8(3+n)} \{2 + 4w + n(1 + 2v + 3v^2 - 4vw + w^2)\} \rho^2 - \chi , \quad (39)$$

$$-3\dot{H}_a - 6H_a^2 = 8\pi Gv\rho + \frac{\kappa^4}{8(3+n)} \{2 - 2v - 6v^2 + 6w + 6vw + n(1 + 2v + 3(v-w)^2)\} \rho^2 - \frac{1}{n}(e - 3\chi) , \quad (40)$$

and

$$\dot{\rho} + 3(1+w)H_a\rho = 0 , \quad (41)$$

$$\dot{e} + 3H_a(e + \chi) + \frac{3n}{4(3+n)}\kappa^4 [(w-v)^2\rho\dot{\rho} + H_a(1+w)(w-v)\rho^2] = 0 . \quad (42)$$

What we want is the effective Friedman equation in the Kaluza-Klein braneworld. For that purpose, a glance of Eq. (38) tells us that we need to know  $e$ , which is a component of the projected Weyl tensor which encodes some information about the bulk geometry. For general  $w$ , Eq. (41) is solved to give

$$\rho = a^{-3(1+w)} , \quad (43)$$

where we have absorbed a constant factor into the scale factor by rescaling it. This is a standard result. Eliminating  $\dot{H}$  and  $H^2$  from Eqs. (38)-(40), we obtain

$$-n\xi = 3\chi - e = 8\pi G \frac{n}{2+n} (3w - 2v - 1)\rho + \kappa^4 \frac{n}{4(3+n)} v (1 - 3w + 3v) \rho^2. \quad (44)$$

Thus the components of the projected Weyl tensor are related to the matter on the brane. Substitution of the above result (44) into the constraint equation for the dark radiation (42) gives the equation

$$\dot{e} + 4H_a e = -8\pi G \frac{n}{2+n} (3w - 2v - 1) H_a \rho - \kappa^4 \frac{n}{4(3+n)} (1 - 3w + 3v) \{v + 3(1+w)(w-v)\} H_a \rho^2. \quad (45)$$

This equation can be integrated easily. Note that the cases  $w = 1/3$  and  $w = -1/3$  need to be considered separately.

First, we consider the generic case  $w \neq 1/3, -1/3$ . The solution is given by

$$e = -\frac{3C}{a^4} + 8\pi G \frac{n}{2+n} \frac{1+2v-3w}{1-3w} a^{-3(1+w)} + \kappa^4 \frac{n}{8(3+n)} \frac{(1-3w+3v)}{(1+3w)} \{v + 3(1+w)(w-v)\} a^{-6(1+w)}, \quad (46)$$

where  $C$  is a constant of integration which can be interpreted as dark radiation [37, 41]. Substituting Eq. (46) into Eq.(38), we obtain the effective Friedman equation

$$H_a^2 = \frac{8\pi G_{\text{eff}}}{3} \rho + A \rho^2 + \frac{C}{a^4}, \quad (47)$$

where the coefficients are given by

$$G_{\text{eff}} = \frac{2(1-3w-nv)}{(2+n)(1-3w)} G, \quad (48)$$

$$A = \frac{2+6w+n(1+2v+3(v-w)^2)}{24(3+n)(1+3w)} \kappa^4. \quad (49)$$

The above equations include the well-known 5D case, corresponding to  $n = 0$  and for which  $G_{\text{eff}} = G$  and  $A = \kappa^4/36$ . By contrast, in higher dimensions, the effective Newton constant that we have defined depends on the equation of state. It means that the Kaluza-Klein braneworld cosmology, provided our assumptions are valid, is different from the conventional Kaluza-Klein cosmology even at low energies. For  $w = 0$ , we have to assume  $nv < 1$  in order to have the positive effective Newton constant. It should be noticed that the effective Newton constant becomes negative in the regime  $0 < nv < 1$  and  $(1-nv)/3 < w < 1/3$ . This indicates some transient instability around the matter-radiation equality.

Before proceeding, however, it is important to note that in addition to the assumption of staticity of the internal space, we have also assumed that the bulk matter can be neglected and that there is no explicit coupling between the matter on the brane and the matter in the bulk. If we relax one or several of these assumptions, the conclusion will be altered.

Now, we consider the radiation-dominated case  $w = 1/3$ . In this case, the solution reads

$$e = -\frac{3C}{a^4} + 8\pi G \frac{2n}{2+n} v \frac{\log a}{a^4} - \kappa^4 \frac{n}{16(3+n)} v (9v - 4) a^{-8}. \quad (50)$$

There appears a logarithmic correction in the above. Substituting Eq. (50) into Eq. (38), we obtain the effective Friedman equation (47) with the coefficients given by

$$G_{\text{eff}} = G \left( 1 - \frac{2n}{2+n} v \log \frac{a}{a_*} \right), \quad (51)$$

$$A = \kappa^4 \frac{12 + n(4 + 9v^2)}{144(3+n)}, \quad (52)$$

where  $a_*$  is a constant of integration corresponding to the dark radiation component  $C$ . Remarkably, the effective Newton constant depends logarithmically on the scale factor. This interesting behavior of the cosmological evolution occurs only during a radiation-dominated stage. Furthermore, depending on the value  $a_*$ , the effective Newton

constant may become negative. This implies that the dark radiation component should be chosen appropriately in order to realize a sensible cosmology on the brane. It may be mentioned that this behavior is in contrast to the case of the Brans-Dicke cosmology in which the effective Newton constant begins to depend on time after pressureless matter starts to dominate. In a sense, one may say that dark matter stabilizes the effective gravitational coupling constant. The relevant constraint comes from nucleosynthesis [42, 43]:

$$\frac{\Delta G_{\text{eff}}}{G_{\text{eff}}} = 0.01_{-0.16}^{+0.20}, \quad (53)$$

at one-sigma confidence level. It is easy to see that this constraint is satisfied for a sufficiently large  $a_*$ .

Finally, we consider the case  $w = -1/3$ . The solution is given by

$$e = -\frac{3C}{a^4} + 8\pi G \frac{n}{2+n}(1+v)a^{-2} + \kappa^4 \frac{n}{12(3+n)}(2+3v)^2 \frac{\log a}{a^4}, \quad (54)$$

where a logarithmic term appears again. Substituting Eq. (54) into Eq. (38), we obtain the coefficients in the effective Friedman equation as

$$G_{\text{eff}} = \frac{2-nv}{2+n}G, \quad (55)$$

$$A = \kappa^4 \frac{6+n(2-6v-9v^2)}{72(3+n)} - \kappa^4 \frac{n}{36(3+n)}(2+3v)^2 \log \frac{a}{a_*}, \quad (56)$$

where  $a_*$  is the constant of integration we mentioned before. In this case, we have a logarithmic scale factor dependence in the coefficient of  $\rho^2$ . This can have some impact at high energy. Therefore, its effect on the inflationary scenario may be interesting.

Apparently, the Kaluza-Klein cosmology we have obtained is different from the conventional Kaluza-Klein cosmology. In particular, the effective Friedman equation depends on the equation of state of the matter explicitly. This result will hold in more general Kaluza-Klein spacetimes. The reason is the following. The constraint equation (18) in the absence of the bulk matter reads

$$\nabla^\mu E_{\mu\nu} = \kappa^4 \nabla^\mu \pi_{\mu\nu}. \quad (57)$$

Hence, in general, the projected Weyl tensor is affected by the energy momentum tensor on the brane. If the brane was isotropic and homogeneous, the matter part would have the additional property that  $\nabla^\mu \pi_{\mu\nu} = 0$ . In our example, this can be seen by setting  $P = Q$  in Eq. (32). The effect of matter would then not appear in Eq. (57). This is related to Birkoff's theorem. Because of spherical-like symmetry, one does not see the details of the matter contents, but rather see the dark radiation which is independently conserved. On the contrary, in the case of Kaluza-Klein braneworlds this no longer happens because of the anisotropy of the brane. In particular,  $\pi_{\mu\nu}$  is no longer conserved,  $\nabla^\mu \pi_{\mu\nu} \neq 0$ . Thus the anisotropic brane will deform the bulk geometry nontrivially. Then the back-reaction of this to the brane will modify the effective Friedman equation. Because of the strong anisotropy of the Kaluza-Klein braneworld, we then expect that the modification of the Friedmann equation will persist even at low energies.

Since the SMS equation is not a closed system, we cannot formulate cosmological perturbation theory without further information. As we noted before, we may regard the SMS equations as giving the “initial data” to solve the bulk geometry. Of course, in general, there would be a singularity in the bulk. However, if one introduces some bulk matter, one may obtain a non-singular geometry in the bulk. With some explicit bulk matter, one can also formulate explicitly the cosmological perturbation theory for the corresponding Kaluza-Klein braneworld. Since there is the possibility of a deviation from Newton's law at low energy, it would be interesting to perform this program explicitly.

#### IV. CONCLUSION

We have considered a  $(d+1)$ -dimensional gravitational system with bulk matter and a  $d$ -dimensional brane, and derived the effective  $d$ -dimensional Einstein equations on the brane. As an application, we have studied the cosmology of Kaluza-Klein braneworld, with  $n$  internal toroidal dimensions. By neglecting the bulk matter and imposing the staticity of the internal space, we have obtained a closed set of equations, from which we have been able to derive the effective Friedmann equation. We have found that the resultant Friedmann equation explicitly depends on the equation of state. In particular, if the universe is dominated by radiation, a resonant contribution to the projected Weyl tensor gives a time variation to the effective gravitational coupling constant. This time dependence disappears



after the radiation-matter equality, which can be interpreted as a stabilization of the Newton constant by the matter on the brane. It should be emphasized that the Kaluza-Klein braneworld cosmology is quite different from the conventional Kaluza-Klein cosmology even at low energy. It is in contrast to the fact that the conventional Friedmann equation can be recovered at low energy in the RS braneworld cosmology. Hence, it is important to see whether Newtonian gravity can be recovered [44] on the Kaluza-Klein braneworld. This question has already been studied in some specific 6D models based on flux compactifications [22, 45].

We have also discussed the method to obtain the bulk geometry from the brane data. It seems always possible to adjust  $E_{\mu\nu}$  so that the braneworld has a static internal space during the cosmic expansion. Of course, in general, the geometry of the bulk will be contrived and it will perhaps contain singularities. However, assuming the presence of matter in the bulk, there is a chance to have a non-singular bulk. Admittedly, it is a non-trivial problem to find an appropriate matter which can stabilize the braneworld without any pathology in the bulk. An alternative approach is to start from known bulk solutions in which a brane is embedded. This method has been used very recently to study the cosmology of Kaluza-Klein branes, with one internal dimension, in 6D bulk solutions of Einstein-Maxwell equations [46, 47].

There are many other issues to be explored. It is interesting to formulate the quantum creation of the Kaluza-Klein braneworld as is done in RS models [48, 49]. It is also important to understand the low energy description of the Kaluza-Klein braneworld [50, 51, 52, 53, 54, 55, 56] and the Kaluza-Klein corrections [57, 58]. It is intriguing to consider born-again scenario in higher dimensions [59, 60]. We leave these issues for future studies.

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